UNCERTAINTY TUTORIAL

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1

INTRODUCTION

Since there is no absolute proof in science, the validity of a theory is generally evaluated by doing many, many experiments, checking to see if the results of each experiment agree with the theory, and modifying the theory as necessary. In order to compare a result to an expected value or to the value from another experiment, it is necessary to know the uncertainty of each experimental result. For example, if you measure the acceleration of gravity to be 9.75 m/s^2 and you wish to compare this to the accepted value of 9.81 m/s^2, this evaluation will depend on the uncertainty of your experimental value. If the uncertainty of your measured value is 0.10 m/s^2, this indicates that your experimental result could actually be anywhere between 9.75 m/s^2 - 0.1 m/s^2 = 9.65 m/s^2 and 9.75 m/s^2 + 0.10 m/s^2 = 9.85 m/s^2. Since this interval includes the accepted value of 9.81 m/s^2, we would say that the experimental result is in agreement with the accepted value. However, if the uncertainty of your measurement is 0.04 m/s^2, then the range for your experimental value is from 9.75 m/s^2 - 0.04 m/s^2 = 9.71 m/s^2 to 9.75 m/s^2 + 0.04 m/s^2 = 9.79 m/s^2. This range does <u>not</u> include the accepted value of 9.81 m/s^2, so in this case the experiment is <u>not</u> in agreement with the accepted value. The uncertainty of an experimental result is usually written as a plus or minus value added to the experimental result. For example, in the first case above, the experimental value would be reported as (9.75 ± 0.10) m/s^2. In the second case, the experimental value would be reported as (9.75 ± 0.04) m/s^2.

If, taking uncertainty into consideration, an experimental result does not agree with a theoretical result, there are two possibilities. The first of these is by far the more common and will almost certainly be the reason that the result of an instructional lab is not in agreement with the theory. That is that there are sources of error present in the experiment that were not accounted for in the estimate of uncertainty. If such an error source is one that could reasonably have been quantified but was simply omitted due to an oversight, the uncertainty calculations should be redone to include that source of error. However, in many cases there are sources of error that can be observed or deduced by the experimenter that are difficult or impossible to quantify. Such errors should <u>always</u> be discussed in the course of evaluating the experimental results. (Note that carelessness or errors in calculation should <u>never</u> be listed as sources of error. These are mistakes that should be corrected, not legitimate sources of error.)

Suppose you perform two experiments to measure the radius of the earth. In the first experiment, you obtain a value of (6300 ± 100) km. In the second experiment, your result is (6350 ±10) km. Is either of these results in agreement with the theoretically accepted value of 6.37 x 10^6 m?

If you answered yes for the first experiment and no for the second, go to

2

Otherwise continue.

In the first example above, the lower limit of the experimental range is 6300 - 100 = 6200 km, and the upper limit is 6300 + 100 = 6400 km. The theoretical value is

$$(6.37 \text{ x } 10^6 \text{ m})\left(\frac{1\,km}{10^3\,m}\right) = 6.37 \text{ x } 10^3 \text{ km, or } 6370 \text{ km.}$$

Since this is within the 6200 - 6400 km range, the experimental value agrees with the theoretical one in this case.

In the second example, the lower limit of the experimental range is 6350 - 10 = 6340 km; the upper limit is 6350 + 10 = 6360 km. The theoretical value of 6370 km is not included within the 6340 - 6360 km range, so in this case the experimental and theoretical values are not in agreement. Notice that even though the result in the second example seems closer to the theoretical value, this result is not acceptable, while the result in the first example is acceptable.

Which of the following experimental results is compatible with a theoretical value for the speed of light of 3.00 x 10^8 m/s?
(a) (3.14 x 10^8 ± 10) m/s
(b) (2.98 ± 0.01) x 10^8 m/s
(c) (2.95 ± 0.10) x 10^8 m/s

If you chose (c), continue to

2

If not, return to

1

2 DETERMINING THE UNCERTAINTY OF MEASUREMENTS

We now address the task of determining the uncertainties of measured quantities, and combining these uncertainties to determine the uncertainty of an experimental result calculated from such measured data. The idea that the uncertainty of individual measured values combine to produce an uncertainty in the result that includes contributions from all the quantities used to obtain that result is called propagation of uncertainty (sometimes erroneously labeled as propagation of error). It should be remembered that what follows is intended as a guide to help you to understand some of the reasoning involved in this process. It should not be viewed as a set of ironclad rules to be blindly followed and which will lead, by rote application of the rules, to the ultimate value of a mystical "uncertainty". There is no one "recipe" that will work for calculating uncertainty in all circumstances. As you work your way through this tutorial, try to understand the logical processes involved, rather than memorizing a set of rules. This should enable you to modify the process as necessary for any specific application. As in all other aspects of scientific inquiry, there is no substitute for logical thought and common sense.

In order to determine the uncertainty of any experimentally determined quantity, we must start by determining the uncertainty of each of the measured elements of the data that will be used to calculate that experimental result. How this uncertainty is determined will depend on the particulars of the data obtained in any particular experiment. Ideally each measurement is repeated some very large number of times, so that a statistical analysis of such values can be used to determine the uncertainty of the measured quantity. However, in instructional labs, time and other constraints often limit the amount of data that can reasonably be obtained. Your lab manual[1] lists the following guidelines for determining the uncertainty of a particular measurement in various circumstances:

1. If more than 50 trials have been taken, represent the uncertainty by the rms deviation.
2. If more than 10 but fewer than 50 trials have been taken, report the uncertainty as the average of the absolute values of the deviations of the individual values.
3. If fewer than 10 measurements have been taken, report the uncertainty as the largest of the individual deviations.
4. If only one or two experimental values are recorded, the observer must make a reasonable estimate of the uncertainty, based on experience with the measuring apparatus and the characteristics of the object being measured.

We will examine each of these cases separately, starting with the last one and working backwards, defining the appropriate terms as we go.

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[1]Cochran, William, *The Physics Lab Manual (I & II)*, page 14, Department of Physics & Astronomy, Youngstown State University, 1992

2a

Uncertainty Value from One or Two Trials

If only one or two measurements of a particular quantity have been made, an estimate of the uncertainty of the measurement must be made based on the properties of the measuring instrument used. For example if a length measurement is made using a meter stick on which the finest divisions represent tenths of a centimeter, then the measurement might be "off" by as much as half the distance between these marks, i.e.

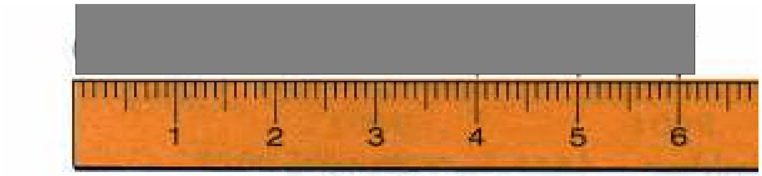
$$\left(\frac{1}{2}\right)\left(\frac{1}{10}cm\right) = \left(\frac{1}{20}cm\right) = .05\,cm$$

That is, the measurement might be high or low by this amount. So for example if we measured the length of a block of wood to be 8.4 cm, we would write this measurement as 8.4 ± .05 m.

If we used a vernier caliper which is capable of making measurements to the nearest hundredth of a centimeter to measure the diameter of a metal rod to be 1.56 cm, the uncertainty of this measurement would be

$$\left(\frac{1}{2}\right)\left(\frac{1}{100}cm\right) = \frac{1}{200}cm = .005\,cm$$

It should be noted here that the uncertainty is not always just one-half the distance between divisions of the measuring instrument. For instance, we might expect that the uncertainty of a measurement of potential made on a meter that can measure to the nearest tenth of a volt would be .05 volts. If the meter gives a stable measurement to the nearest tenth of a volt and we have no reason to believe that the calibration of the meter is off or that the readings for a particular measurement vary over time, then .05 volts would be a reasonable estimate of the uncertainty of a measurement made with this meter. However, if we notice that the meter reading does not stabilize, but rather fluctuates over time, say between values of 10.3 volts and 11.5 volts, then we should record the potential value as 10.9 volts (the value halfway between the two extremes) with an uncertainty of 0.6 volts (half the range of the variation). That is, this measurement would be recorded as 10.9 ± 0.6 volts. As noted before, there is no definite rule for determining uncertainty in all cases. We must make the most reasonable estimate possible, based on all the relevant information available.



1. Give the length of the gray bar above, and its associated uncertainty, as measured by the ruler shown. (The numbered markings on the ruler are centimeters, and the diagram has been enlarged to make it easier to read.)

2. You are using a multimeter to measure the resistance of a commercial resistor. The meter gives readings to the nearest one hundredth of an ohm, and when connected across the resistor,
the meter gives readings that fluctuate between 346.59 ohms and 351.08 ohms. Give a value for the measured resistance and its associated uncertainty.

If you answered 6.2 ± .05 cm for the first question and 348.84 ± 2.25 ohms for the second question, go to

2b

Otherwise, continue.

1. The numbers on the ruler represent centimeters, so the markings in between the centimeter marks represent tenths of a centimeter. By inspection, we see that the length of the bar is 6.2 cm. The uncertainty of the measurement is half the interval between these finest markings,
i.e.

$$\left(\frac{1}{2}\right)\left(\frac{1}{10}cm\right) = \left(\frac{1}{20}\ cm\right) = .05cm$$

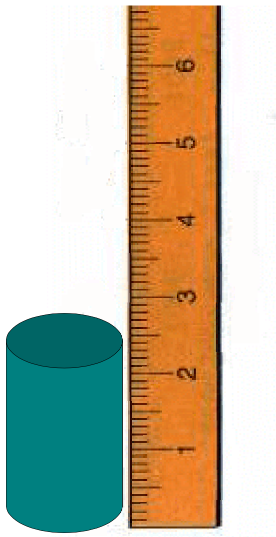
So the measurement is written as 6.2 ± .05 cm.

2. Since the reading fluctuates, we take the average value of the two extremes:

$$\left(\frac{346.59\,ohms + 351.08\,ohms}{2}\right) = 348.84\,ohms$$

The uncertainty is one half the range of the fluctuation:

$$\left(\frac{1}{2}\right)\left(351.08\,ohms = 346.59\,ohms\right) = 2.25\,ohms$$



So the measusrement is recorded as 348.84 ohms ± 2.25 ohms

1. Use the ruler at left to give a measurement, with associated uncertainty, of the height of the cylinder shown.

2. What measurement and uncertainty should be recorded for the current measured by an ammeter, if the reading on the ammeter fluctuates between 8.9 amps and 10.1 amps?

If you answered 2.4 ± .05 cm for the first question and 9.5 ± 0.6 amps, continue to

2b

Otherwise, return to

2a

2b Uncertainty Value from a Small Number of Trials

If more than one measurement of a physical quantity is made, the value of the quantity that is used should be the average of the measured values. (Note that all of the raw data should be recorded before calculating the average.) The uncertainty of the measurement is based on the deviation of the individual measurements from the average. The deviation is calculated for each measurement by first calculating the average of the measurements, then finding the difference between each measurement and the average. For example, suppose four measurements are taken for a particular current, and the values of those measurements are: 3.5 amps, 2.9 amps, 3.1 amps, and 3.2 amps. The average of the four values (i.e. the mean value) is

$$\frac{(3.5 + 2.9 + 3.1 + 3.3)\,amps}{4} = 3.2\,amps$$

and the deviation for each of these measurements is

(3.5 - 3.2) amps = 0.3 amps
(2.9 - 3.2) amps = -0.3 amps
(3.1 - 3.2) amps = -0.1 amps
(3.3 - 3.2) amps = 01 amps

This data would normally be collected in a table like the one below.

	Current reading (amps)	Deviation (amps)
Trial #1	3.5	0.3
Trial #2	2.9	-0.3
Trial #3	3.1	-0.1
Trial #4	3.3	0.1
Mean (average)	3.2	

Since there are so few trials available in this case, the uncertainty here should be reported simply as the absolute value of the largest value of the deviation, i.e. 0.3 amps. (See rule #3 above or on page 14 of your lab manual, which applies when there are fewer than 10 trials.) So the experimental value of the current in this case would be reported as 3.2 ± 0.3 amps.

If data for more than 10 trials for a particular measurement are available, but there are still not enough trials to make a more rigorous statistical analysis feasible (i.e. between 10 and 50 trials according to rule #2 above and in your lab manual), a reasonable value of the uncertainty can be found by calculating the average of the absolute values of the individual deviations. For example, suppose the table below represents the data for 14 trials for a particular distance measurement.

	Length in cm	Deviation in cm	Abs. Val. of Dev. (cm)
Trial #1	45.3	2.1	2.1
Trial #2	41.2	-2	2
Trial #3	42.8	-0.4	0.4
Trial #4	46.0	2.8	2.8
Trial #5	39.9	-3.3	3.3
Trial #6	43.7	0.5	0.5
Trial #7	44.4	1.2	1.2
Trial #8	45.1	1.9	1.9
Trial #9	40.6	-2.6	2.6
Trial #10	41.2	-2	2
Trial #11	45.6	2.4	2.4
Trial #12	43.5	0.3	0.3
Trial #13	44.9	1.7	1.7
Trial #14	40.2	-3	3
Mean (Average)	43.2	0.0	1.9

Notice that the mean value of the deviation is not useful since, by definition, it will always be zero. (There will always be the same amount of total deviation above the mean value as below.) But what we want to know is, on average, how much do the measured values for the different trials, deviate from the mean value, not whether they are above or below the mean. So the average of the absolute values of the deviations gives us a reasonable estimate of the uncertainty of the measurement.

The results of these measurements would be reported as a length of 43.2 ± 1.9 cm (the mean value of the measurement ± the mean value of the absolute values of the deviations).

One additional consideration should be mentioned here. Sometimes in the course of performing several trials for a given measurement, one or more trials will yield a measurement that is clearly out of line with the rest of the measurements. In such a case, the aberrant value or values should be discarded and the analysis described above should be done with the remaining values. (All measured values should be included as part of the raw data, and a note describing the reasons for discarding any particular data should be included.) For example, suppose 12 trials for the measurement of a particular force yield the following values, in Newtons:

2.1 2.4 1.9 2.2 2.3 5.6 2..4 2.1 2.0 2.4 2.5 2.2

The value of 5.6 is clearly not consistent with the values of the other 11 trials, so this value should be disregarded and the force measurement should be taken as the average of the remaining 11 trials, with an uncertainty equal to the average of the absolute values of the deviations of those 11 trials.

The following list shows the results of 15 measurement trials for a mass value in a particular experiment. What value of the mass should be reported, and what is the associated uncertainty?

65.3g 62.1g 66.4g 61.9g 60.8g 64.4g 59.9g 63.7g
64.5g 88.1g 62.8g 63.4g 65.0g 61.2g 62.9g

If your answer is 63.2 ± 1.5 g, go to

2c

Otherwise continue.

Before analyzing the given data, we first notice that the value 88.1 g is inconsistent with the rest of the data, so we discard that value. We then organize the rest of the values into a table and calculate the mean value.

	Mass in g	Deviation in g	Abs. Val. of Dev. in g
Trial #1	65.3	2.1	2.1
Trial #2	62.1	-1.1	1.1
Trial #3	66.4	3.2	3.2
Trial #4	61.9	-1.3	1.3
Trial #5	60.8	-2.4	2.4
Trial #6	64.4	1.2	1.2
Trial #7	59.9	-3.3	3.3
Trial #8	63.7	0.5	0.5
Trial #9	64.5	1.3	1.3
Trial #10	62.8	-0.4	0.4
Trial #11	63.4	0.2	0.2
Trial #12	65.0	1.8	1.8
Trial #13	61.2	-2	2
Trial #14	62.9	-0.3	0.3
Mean (Average)	63.2	0.0	1.5

Taking the average of the 14 trials in the second column, we obtain 63.2 g. The deviation for each trial is then determined by subtracting 63.2 g from the individual values for each trial. These values appear in the third column. The absolute values of the deviations appear in the last column. The average of absolute values of the deviations is 1.5 g, as seen at the bottom of the last column. The value for the mass is reported as the mean value of the mass for the 14 trials ± the mean of the absolute value of the deviation, i.e. 63.2 ± 1.5 g.

The following is a list of the measured values of the potential, in volts, taken from 12 trials in a given experiment. Find the voltage value that should be reported, along with its associated uncertainty.

16.1 15.9 16.0 19.5 16.2 16.3 16.0 15.8 16.4 16.1 16.2 15.9

If you answered 16.1 ± 0.1 volts, continue to

2c

Otherwise, return to

2b

2c

Uncertainty Value from a Large Number of Trials

If a large number of trials (greater than 50 according to rule #1 above and your lab manual) are performed for a given measurement, then a statistical analysis is feasible and the uncertainty of the measurement should be calculated as the rms (root mean squared or standard) deviation from the mean. In order to calculate the rms deviation for a set of data, we proceed as before, calculating the mean value, and then the deviation of each value from the mean. But instead of using the absolute value of each deviation value, we instead calculate the square of each deviation value and take the average of those squared values. Like taking the absolute value, this has the effect of eliminating the negative signs since both positive and negative values of the deviation yield positive values for the deviation squared. But using this technique also gives greater weight in the average to values that are further from the mean. After taking the average of the values of the deviation squared, we take the square root of the result in order to obtain the standard deviation, which becomes our desired estimate of the uncertainty. The Greek letter sigma (σ) is traditionally used to represent the standard deviation and the process just described can be written mathematically. The formula for calculating the standard deviation for n trials of measurement of the variable x, the mean value of which is $\bar{x}$, is

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

The process is illustrated here using a data set from 2b. As previously stated, this process would not normally be used for such a small sample; however we use it here to make it easier to see the details of the procedure. The process is the same for a larger sample. In practice, a computer is usually used to calculate the standard deviation. This function is included in most graphing and statistical programs. (A caution is in order here. When using a computer to calculate the standard deviation, make sure you know for what variable it is being calculated. For example, if you do a linear fit for a set of graphed data and you wish to obtain the standard deviation of the slope of the line thus obtained, you may, if you are not careful, obtain instead the standard deviation for the distribution of values on either the x or the y axis. Always check carefully to see what distribution is being used for the calculation.)

Consider the data shown in the table below, for 14 trials of a measurement of length. The mean value of the length has been calculated, as before to be 43.2 cm, and the deviation for each of the measured values is shown in the third column. But now the last column, instead of containing the absolute value of the deviation shows the square of the value of the deviation for each entry.

	Length (cm)	Deviation (cm)	Deviation^2 (cm)^2
Trial #1	45.3	2.1	4.4
Trial #2	41.2	-2	4.0
Trial #3	42.8	-0.4	0.2
Trial #4	46.0	2.8	7.8
Trial #5	39.9	-3.3	10.9
Trial #6	43.7	0.5	0.3
Trial #7	44.4	1.2	1.4
Trial #8	45.1	1.9	3.6
Trial #9	40.6	-2.6	6.8
Trial #10	41.2	-2	4.0
Trial #11	45.6	2.4	5.8
Trial #12	43.5	0.3	0.1
Trial #13	44.9	1.7	2.9
Trial #14	40.2	-3	9.0
Mean (Average)	43.2	0.0	4.4

The mean value of the values in the deviation squared column is computed (simply by adding all the values and dividing the result by 14) and recorded at the bottom of that column. The standard deviation is then calculated by taking the square root of the mean value of the deviation squared.

That is, $\sigma = \sqrt{4.4\,cm^2} = 2.1\,cm$

Find the mean and the standard deviation of the mass for the distribution shown below.

	Mass (g)
Trial #1	29.5
Trial #2	28.3
Trial #3	26.1
Trial #4	30.2
Trial #5	24.9
Trial #6	29.0
Trial #7	25.8
Trial #8	30.1
Trial #9	27.9
Trial #10	26.4
Trial #11	28.0
Trial #12	27.5
Trial #13	25.1
Trial #14	26.7

If you answered 27.5 $\pm$ 1.7 g, go to

3

Otherwise continue.

First calculate the mean value of the mass by adding all the entries in the mass column and dividing by 14 (the number of entries). This yields a value of 27.5 g as shown at the bottom of the mass column below. Then subtract this mean value from each of the entries in the mass column to obtain the deviation for each entry. These are shown in the third column below. Next square each of these values and record the squared values in the fourth column (deviation)^2. Then average the values in the deviation squared column by adding the entries and dividing by 14. The value obtained is 2.9, as shown at the bottom of the that column. The standard deviation is the square root of this value. That is, $\sigma = \sqrt{2.9\,g^2} = 1.7\,g$

So the experimental mass value would be recorded as 27.5 ± 1.7 g

	Mass (g)	Deviation (g)	Deviation^2 (g^2)
Trial #1	29.5	2	4.0
Trial #2	28.3	0.8	0.6
Trial #3	26.1	-1.4	2.0
Trial #4	30.2	2.7	7.3
Trial #5	24.9	-2.6	6.8
Trial #6	29.0	1.5	2.3
Trial #7	25.8	-1.7	2.9
Trial #8	30.1	2.6	6.8
Trial #9	27.9	0.4	0.2
Trial #10	26.4	-1.1	1.2
Trial #11	28.0	0.5	0.3
Trial #12	27.5	0	0.0
Trial #13	25.1	-2.4	5.8
Trial #14	26.7	-0.8	0.6
Mean (Average)	27.5	0.0	2.9

Find the mean and the standard deviation of the current for the distribution shown below.

	Current (amps)
Trial #1	2.13
Trial #2	1.98
Trial #3	1.90
Trial #4	2.07
Trial #5	2.22
Trial #6	1.99
Trial #7	2.16
Trial #8	1.88
Trial #9	1.92
Trial #10	2.15
Trial #11	2.09

If you obtained 2.04 ± 0.12 amps, continue to

3

Otherwise, return to

2c

3

PROPAGATION OF UNCERTAINTY
(Combining Uncertainties of more than one Measurement)

When obtaining experimental results, it is usually necessary to combine several different measured quantities using some mathematical rule. Whenever two or more measurements are combined to obtain an experimental value of another physical quantity, the uncertainty of the measurements must also be combined appropriately to obtain the uncertainty of the calculated result. Here we develop some rules for combining the uncertainties of experimental measurements. We'll start by defining some symbols and terminology that will be useful in discussing the combination of uncertainties for several different experimental variables.

For each experimentally measured variable, an associated uncertainty will be defined; we will use the lower case Greek delta (δ) in front of a variable to denote the uncertainty for that variable. So, for example, the uncertainties for the variables x, y and z would be represented by δx, δy and δz, respectively.

3a

Calculating Fractional Uncertainty and % Uncertainty

Often it is useful to obtain not just a numerical value of the uncertainty, but the % uncertainty, that is the per cent that the uncertainty is of the average measured value of the associated quantity. The per cent uncertainty of a variable is obtained by dividing the uncertainty for that variable by the average value of the variable and multiplying by 100.

$$\% \; uncertainty \; in \; x = \left(\frac{\delta x}{\bar{x}}\right)(100)$$

Notice that the quantity $\frac{\delta x}{\bar{x}}$ is the fractional uncertainty of the variable x and the % uncertainty is obtained (as with other percentages) by multiplying the fractional uncertainty by 100.

Suppose the values shown below were obtained from several trials of a mass measurement in an experiment. What is the % uncertainty for the mass?

54.8 g   49.7 g   52.2 g   54.1 g   53.5 g   79.9 g   50.6 g   51.5 g

If you obtained 5.0 %, go to

3b

Otherwise continue.

First discard the measured value of 79.9 g since this value is clearly out of line with the rest of the data.

Next find the average value for the rest of the data:

$$\overline{m} = \frac{(54.8 + 49.7 + 52.2 + 54.1 + 53.5 + 50.6 + 51.5)\,g}{7} = 52.3\,g$$

Now calculate the deviation for each of the mass values by subtracting the average value from each individual value. These values are collected in the table below.

	Mass (g)	Deviation (g)
Trial #1	54.8	2.5
Trial #2	49.7	-2.6
Trial #3	52.2	-0.1
Trial #4	54.1	1.8
Trial #5	53.5	1.2
Trial #6	50.6	-1.7
Trial #7	51.5	-0.8

Since there are fewer than 10 trials, the uncertainty is taken as the absolute value of the largest of the individual deviations, in this case 2.6 g.

To find the % uncertainty, we take the uncertainty (2.6 g) divided by the average value (52.8 g), and multiply by 100.

$$\%\, uncerta\text{int}\, y\ in\ m = \left(\frac{2.6\,g}{52.3\,g}\right)(100) = 5.0\%$$

Find the % uncertainty for the force, given the following set of data for 12 trials.
(Don't forget to use the appropriate rule for finding uncertainty for the number of trials shown.)

16.2 N	15.1 N	37.3 N	14.5 N	16.6 N	17.0 N
17.8 N	15.0 N	14.9 N	16.9 N	15.7 N	17.1 N

If you answered 5.6 %, go to

3b

If not, see the solution below.

Start by throwing out the value of 37.3 N since this value is incompatible with the rest. Then calculate the average of the rest of the values. The value obtained, 16.1 N, appears at the bottom of the first column in the table below. Next compute the deviation for each entry by subtracting 16.1 N from each of the values in the first column. Then find the absolute value for each of the entries in the deviation column. Finally take the average of the absolute values to obtain 0.9 N (bottom of last column) as the value of the uncertainty in the force measurement. (We use the average of the absolute values of the deviations since there are more than 10 but less than 50 trials.)

	Force (N)	Deviation (N)	Abs. Val. of Dev. (N)
Trial #1	16.2	0.1	0.1
Trial #2	15.1	-1	1
Trial #3	14.5	-1.6	1.6
Trial #4	16.6	0.5	0.5
Trial #5	17.0	0.9	0.9
Trial #6	17.8	1.7	1.7
Trial #7	15.0	-1.1	1.1
Trial #8	14.9	-1.2	1.2
Trial #9	16.9	0.8	0.8
Trial #10	15.7	-0.4	0.4
Trial #11	17.1	1	1
Mean (Average)	16.1	0.0	0.9

So we have $\overline{F}$ = 16.1 N and δF = 0.9 N, and

$$\% \, uncertainty = \left(\frac{\delta F}{\overline{F}}\right)(100) = \left(\frac{0.9\,N}{16.1\,N}\right)(100) = 5.6\%$$

If you're still having difficulty calculating uncertainties, return to

2

If you're having difficulty calculating % uncertainty, go back to

3a

Otherwise continue.

3b

Uncertainty of a Sum or Difference

If a new quantity, say x, is obtained by adding or subtracting two other quantities, say y and z, it is intuitively apparent that the uncertainty in x will be the sum of the uncertainties in y and z.

For example, suppose you measure the length of the lab table using two meter sticks, by placing the zero end of the first meter stick flush with one end of the table and then placing the zero end of the second meter stick so as to coincide with the 1.0 m end of the first meter stick. You then take a reading from the second meter stick at the far end of the table and add this reading to the 1.0 m from the first meter stick. There will be an uncertainty associated with the placement of the zero end of the second meter stick, and an additional uncertainty associated with the reading of the far end of the table on the second meter stick. These two uncertainties should be added in order to obtain the uncertainty of the final measurement. Since this measurement can be thought of as a measurement of one meter with the first meter stick, added to the appropriate measurement from the second stick, we see that the uncertainty of a sum of variables is equal to the sum of the uncertainties for each variable. If you remember that it is only the absolute value of the uncertainty of a measurement that matters, not the sign, it is not difficult to see that the rule of adding uncertainties works with subtraction of variables as well. (There is a somewhat more rigorous derivation of this rule using calculus in the last section of this tutorial for those who are interested.) We can write this rule mathematically as follows:

$$\text{For} \quad x = \bar{y} + \bar{z} \quad \text{or} \quad x = \bar{y} - \bar{z}$$

$$\delta x = \delta y + \delta z$$

(The symbols $\bar{y}$ and $\bar{z}$ for the average values of y and z have been used here since it is the average of all the measured values that would actually be used for each variable.)

Example 1:

If we wish to add the two length measurements, 48.5 cm ± 1.2 cm and 97.2 cm ± 0.4 cm, we would first add the measurements, 48.5 cm + 97,2 cm, to obtain 145.7 cm, then add the two uncertainties, 1.2 cm + 0.4 cm = 1.6 cm. So the final result would be reported as 145.7 ± 1.6 cm. (In this example, the process of finding the uncertainties of each of the individual measurements has been previously completed.)

Example 2:

Subtract 14.9 ± 0.2 g from 67.1 ± 1.0 g

67.1 g - 14.9 g = 52.2 g

0.2 g + 1.0 g = 1.2 g

So the result is 52.2 ± 1.2 g

Add 1.7 ± 0.3 N and 4.6 ± 0.2 N

Subtract 34.21 ± .04 m from 67.90 ± .15 m

If you obtained 6.3 ± 0.5 N and 33.69 ± .19 m, continue to

3c

Otherwise return to

3b

3c

Uncertainty of a Product or Quotient

If two variables are multiplied or divided to obtain a third value, the fractional uncertainties of the factors (instead of just the uncertainties) must be added to obtain the uncertainty of the product or quotient.

Mathematically,

If $\quad x = (\bar{y})(\bar{z}) \quad$ or $\quad x = \dfrac{\bar{y}}{\bar{z}}$

$$\frac{\delta x}{x} = \frac{\delta y}{\bar{y}} + \frac{\delta z}{\bar{z}}$$

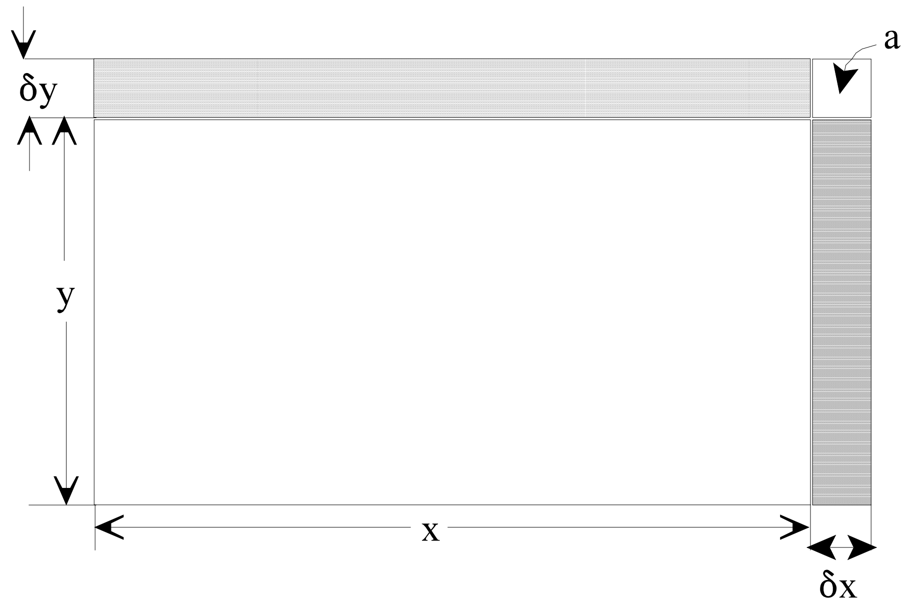
To obtain the % uncertainty from the fractional uncertainty, simply multiply by 100.

To obtain the uncertainty from the fractional uncertainty, multiply by the value of the variable. i.e.

$$\delta x = x\left(\frac{\delta y}{\bar{y}} + \frac{\delta z}{\bar{z}}\right)$$

The validity of the rule for the multiplication of variables can be illuminated by a pictorial representation: (© May 2001, Douglas A. Fowler and Diana M. Ludwig)

Consider the three variables, x, y and A. We interpret x as the measured length and y as the measured width of the rectangle pictured below. δx and δy are the uncertainties in these measurements respectively.



It is apparent from the picture that the total uncertainty, δA, in the calculated area (A = x y) is given by the areas of the shaded, thin rectangles plus the small rectangle with area a; that is

$$\delta A = y\,\delta x + x\,\delta y + a$$

When δx and δy are small, the small area a becomes negligible and the equation above becomes simply

$$\delta A = y\,\delta x + x\,\delta y$$

Dividing both sides of the above equation by xy, we obtain

$$\frac{\delta A}{x\,y} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

or, since A = xy,

$$\frac{\delta A}{A} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

This is the uncertainty rule for multiplication stated above (with the variables A, x, and y substituted for x, y and z respectively.)

<u>Exercise 1</u>:
The experimental values of the mass, m, and acceleration, a, of an object are given by 56.9 ± 1.40 g and 63.4 ± 10.1 cm/s^2, respectively. Use Newton's 2nd law, F = m a, to determine the experimental value of the force in dynes, including its uncertainty. Also give the % uncertainties for m, a, and F

If you answered 3610 ± 660 dynes and 2.46 %, 15.9 % and 18.3 % for m, a and F respectively, skip to Exercise 2
If not, continue.

First compute the fractional uncertainty for the mass and acceleration by dividing the uncertainty by the experimental value for each one.

$$\frac{\delta m}{\overline{m}} = \frac{1.40\,g}{56.9\,g} = .0246 \qquad\qquad \frac{\delta a}{\overline{a}} = \frac{10.1\,cm/s^2}{63.4\,cm/s^2} = .159$$

Notice that the units cancel out here so that the fractional uncertainty has no units.
(This is <u>not</u> true of the uncertainty.)

The fractional uncertainty of the force can now be computed as the sum of the fractional uncertainties of the mass and the acceleration.

$$\frac{\delta F}{F} = \frac{\delta m}{\overline{m}} + \frac{\delta a}{\overline{a}} = .0246 + .159 = .183$$

The experimental value of the force is computed by multiplying the experimental values of mass and acceleration.

$$F = \overline{m}\,\overline{a} = (56.9\,g)(63.4\,cm/s^2) = 3607\,dynes$$

The uncertainty for the force can now be computed by multiplying the value of F just computed by the fractional uncertainty of the force.

$$\delta F = \left(\frac{\delta F}{F}\right) F = (.183)(3607\,dynes) = 660\,dynes$$

Hence the experimental value of the force would be recorded as 3610 ± 660 dynes.
(Values have been rounded to 3 significant figures, to match the precision of the original data.)

The % uncertainty for each variable is computed simply by multiplying the fractional uncertainty by 100.

% uncertainty for m = (.0246) (100) = 2.46 %
% uncertainty for a = (.159) (100) = 15.9 %
% uncertainty for F = (.183) (100) = 18.3 %

Exercise 2:
In an experiment designed to determine the spring constant for a certain spring, the experimental value of the force, F, is found to be 2.43 ± .66 N, for a displacement, Δx, of 27.3 ± 0.8 cm. Using the formula k = F / Δx, find the experimental value of the spring constant, k, including the associated uncertainty. Also determine the % uncertainty for each variable.

If you answered 8.90 ± 2.68 N/m, and 27.2 %, 2.9 % and 30.1 %, for F, Δx and k respectively, go to

3d

If not, continue.

Before making any calculations, convert the units of Δx from cm to m, by dividing by 100, to make them compatible with the units of force.

$$\Delta x = .273 \pm .008 \text{ m}$$

Next compute the fractional uncertainties for F and Δx.

$$\frac{\delta F}{\overline{F}} = \frac{.66\,N}{2.43\,N} = .272 \qquad \frac{\delta(\Delta x)}{\overline{\overline{\Delta x}}} = \frac{.008\,m}{.273\,m} = .029$$

The fractional uncertainty in k is equal to the sum of these.

$$\frac{\delta k}{k} = \frac{\delta F}{\overline{F}} + \frac{\delta(\Delta x)}{\overline{\overline{\Delta x}}} = .272 + .029 = .301$$

The experimental value of k is determined by dividing the experimental value of F by that of Δx.

$$k = \frac{\overline{F}}{\overline{\overline{\Delta x}}} = \frac{2.43\,N}{.273\,m} = 8.90\frac{N}{m}$$

The uncertainty in k is computed by multiplying this value of k by its fractional uncertainty.

$$\delta k = \left(\frac{\delta k}{k}\right) k = (.301)\left(8.90\frac{N}{m}\right) = 2.68\frac{N}{m}$$

So the experimental value of k is reported as 8.90 ± 2.68 N/m.

The % uncertainties are computed by multiplying the appropriate uncertainties by 100.

% uncertainty in F = (.272) (100) = 27.2 %
% uncertainty in Δx = (.029) (100) = 2.9 %
% uncertainty in k = (.301) (100) = 30.1 %

3d Uncertainty of a Quantity Raised to a Power

The rules for finding the uncertainty for a variable raised to a power can be obtained from the multiplication and division rules. For example, to find the uncertainty for the square of some variable x, we make use of the fact that x^2 is simply x times x. So the fractional uncertainty of x^2 will be equal to the sum of the fractional uncertainty in x and the fractional uncertainty in x again. That is the fractional uncertainty in x^2 is equal to twice the fractional uncertainty in x. We can write this symbolically as follows:

$$\frac{\delta(x^2)}{x^2} = \frac{\delta x}{\bar{x}} + \frac{\delta x}{\bar{x}} = 2\left(\frac{\delta x}{\bar{x}}\right)$$

If the power is 3 or 4 or n, we simply add the fractional uncertainty that number of times instead of twice. So the general rule for finding the fractional uncertainty of x^n is given by

$$\frac{\delta(x^n)}{x^n} = n\left(\frac{\delta x}{\bar{x}}\right)$$

If the variable is raised to a negative power, we need only remember that x^{-n} is the same as $1/x^n$. That is, multiplying by x^{-n} is the same as dividing by x^n. Recall from the last section that the uncertainty rule for division is the same as that for multiplication, i.e. in either case we obtain the fractional uncertainty of the result by adding the individual fractional uncertainties of all the variables that are multiplied or divided to obtain the result. So if we replace the n in the above equation with the absolute value of n, this rule works for both positive and negative n.

$$\frac{\delta(x^n)}{x^n} = |n|\left(\frac{\delta x}{\bar{x}}\right)$$

It should be noted that n need not be an integer for this analysis to work, so if n is 1/2 or 1/3 or .004, the same rule for finding the uncertainty applies.

Example 3:
If we wish to find the uncertainty in p^3, where p = 57.9 ± 2.40 kg m/s, we first find the fractional uncertainty in p, then multiply it by 3. This will be the fractional uncertainty in p^3.

$$\frac{\delta(p^3)}{p^3} = 3\left(\frac{\delta p}{p}\right) = 3\left(\frac{2.40\,kg\,m/s}{57.9\,kg\,m/s}\right) = .124$$

The % uncertainty in p^3 is (.124)(100) = 12.4 %.
The value of p^3 is $(57.9 \text{ kg m/s})^3 = 1.94 \times 10^5 \text{ (kg m/s)}^3$

The uncertainty in p^3 is then found by multiplying the fractional uncertainty by the value of $\overline{p^3}$.

$$\delta p^3 = \left(\frac{\delta p^3}{\overline{p^3}}\right) p^3 = (.124)\left[1.94 \text{ x } 10^5 \text{ (kg m/s)}^3\right] = 2.41 x 10^4 \left(kg\,m/s\right)^3$$

It may not be necessary to complete the last step since it is often the percent uncertainty that is actually desired.

Example 4:
In an oscillating spring experiment the experimentally measured values of the mass, m, and the spring constant, k, are determined to be 56.0 ± 1.3 g and 7.8 ± 0.4 N/m respectively. Find the experimental value of the period, T, with its associated uncertainty, using the formula:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

First find the fractional uncertainties for m and k.

$$\frac{\delta m}{\overline{m}} = \frac{1.3\,g}{56.0\,g} = .023 \qquad\qquad \frac{\delta k}{\overline{k}} = \frac{0.4\,N/m}{7.8\,N/m} = .051$$

To obtain the fractional uncertainty of the quotient m/k, we take the sum of these two fractional uncertainties. We then multiply that result by one-half (to account for the square root or one-half power). Notice that the constant, 2π, does not contribute to the uncertainty, since it isn't a measured value.

$$\frac{\delta T}{T} = \frac{1}{2}\left(\frac{\delta m}{\overline{m}} + \frac{\delta k}{\overline{k}}\right) = \frac{1}{2}\left(.023 + .051\right) = .037$$

The experimental value of T is given by

$$T = 2\pi\sqrt{\frac{.056\,kg}{7.8\,N/m}} = .532\,s$$

(Note that the units of mass have been converted to kg in order to match with the units of k.)

So the uncertainty in T is given by (.037)(.532 s) = .020 s, and the experimental value of T is given by .532 ± .020 s. (The % uncertainty for T is (.037)(100) = 3.7 %.

<u>Exercise 3</u>:

Find the experimental value of the force from the formula, $F = \dfrac{mv^2}{r}$, and the data below.

m = 234 ± 1.0 g   v = 560 ± 12 cm/s   r = 42.4 ± 1.6 cm

If you answered 1.73 x 10^6 ± 1.45 x 10^5 dynes, go to

3e

If not, continue.

First find the fractional uncertainties of m, v and r.

$$\frac{\delta m}{\overline{m}} = \frac{1.0\,g}{234\,g} = .0043 \qquad \frac{\delta v}{\overline{v}} = \frac{12\,cm/s}{560\,cm/s} = .021 \qquad \frac{\delta r}{\overline{r}} = \frac{1..6\,cm}{42.4\,cm} = .038$$

The fractional uncertainty of v is multiplied by 2 to account for the square of that factor, then the fractional uncertainties are added.

$$\frac{\delta F}{F} = \frac{\delta m}{\overline{m}} + 2\frac{\delta v}{\overline{v}} + \frac{\delta r}{\overline{r}} = .0043 + 2\left(.021\right) + .038 = .084$$

The experimental value of F is

$$F = \frac{mv^2}{r} = \frac{\left(234\,g\right)\left(560\,cm/s\right)^2}{42.4\,cm} = 1.73x10^6\ dynes$$

The uncertainty of F is

$$\delta F = \left(\frac{\delta F}{F}\right)F = (.084)\left(1.73x10^6\right) = 1.45x10^5\ dynes$$

So the experimental value of F is reported as 1.73 x 10^6 ± 1.45 x 10^5 dynes

(The per cent uncertainty for F is (.084)(100) = 8.4 %.)

If you are still having difficulty with these calculations, return to

3a

If you feel confident with these calculations, continue to

3e

3e

Uncertainty of the log of a quantity

There is no easy way, without calculus, to understand the rule for finding the uncertainty when a log function is involved. So the rule is merely stated here without justification. (For the derivation, using calculus, see the appendix.)

$$\text{If } x = \log y$$

$$\delta x = \frac{\delta y}{\bar{y}}$$

So, for instance, if we have the experimentally measured value, $\lambda = 1.4 \pm .05$ m, and we wish to find the uncertainty in log λ, we would simply divide the uncertainty in λ by the average value of λ.

$$\delta(\log \lambda) = \frac{\delta \lambda}{\bar{\lambda}} = \frac{.05m}{1.4m} = .036$$

The value of log λ is log(1.4) = 0.15, so the fractional uncertainty of λ is

$$\frac{\delta(\log \lambda)}{\log \lambda} = \frac{.036}{0.15} = 0.24$$

and the % uncertainty for log λ is (0.24)(100) = 24 %.
(Notice that there are no units on the logs of any function.)

Find the uncertainty and the per cent uncertainty for log T if the experimental value of T is given by 4.67 ± .03 s.

If you answered .0064 and .96 %, go to

4

Otherwise continue.

The uncertainty of log T is given by $\delta(\log T) = \frac{\delta T}{T} = \frac{.03s}{4.67s} = .0064$

Log T = log(4.67) = .669, so the fractional uncertainty for T is $\frac{\delta(\log T)}{\log T} = \frac{.0064}{669} = .0096$

The % uncertainty for log T is (.0096)(100) = .96 %.

Find the uncertainty and per cent uncertainty for log k if the experimental value of k is given by 11.5 ± 1.2 N/m.

If you answered .10 and 9.8 % continue.
If not return to

3a

Often in the course of an experiment, a graph is made of either one variable vs. another or of the log of one variable vs. the log of the other. A linear fit for the data points on the graph is constructed using the least squares procedure. (This can be done by hand using the description in your lab manual or it can be obtained easily using most any graphing program.) The slope of the line so obtained may be used in conjunction with other variables in a calculation to obtain some other experimental result. In that case the standard deviation of the slope should also be computed (again either by the process outlined in your lab manual or by use of a graphing program) and the slope should be treated the same as any other variable, using the standard deviation of the slope as the uncertainty of the slope and applying the rules discussed in this tutorial. When using a graphing program, make sure that you do, in fact, determine the standard deviation <u>of the slope</u>. Some programs will automatically calculate the standard deviation for the data plotted on one or the other of the axes, if you don't specifically request something else. This is <u>not</u> the same as the standard deviation of the slope.

4

Evaluating Results: % Uncertainty and % Difference

As stated earlier, we calculate the uncertainty of an experimental result in order to allow a meaningful evaluation of the agreement of that experimental result with an “accepted” value established by existing theory, or with the results of other experiments. We usually do this by comparing the % uncertainty of the experimental result either to the % error for that result, or to the % difference between the results of two or more different experiments.

If a theoretically accepted value for the experimentally measured quantity is available, we compute the % error for our measured value, by taking the difference between the experimental value and the accepted value, dividing by the accepted value, and multiplying by 100.

$$\%\,error = \frac{\left| accepted\ value - experimental\ value \right|}{accepted\ value}(100)$$

(The absolute value is used here because it is the difference between these two quantities that is needed. Which is larger than the other is not important.)

For example, if we measure the acceleration of gravity to be 9.75 m/s^2, we can compare this to the accepted value of 9.81 m/s^2 by computing the % error as follows:

$$\%\,error = \frac{\left| 9.81 m/s^2 - 9.75 m/s^2 \right|}{9.81 m/s^2}(100) = .61\%$$

If the % uncertainty for our experimental value is greater than .61 %, then that uncertainty “covers” the error, and we can say that our experimental value is in agreement with the theoretical value. If the % uncertainty is less than .61 %, then there is a discrepancy between the experimental value and the acceptable value that cannot be accounted for by uncertainty in measurement. This means either that there is some source of uncertainty that has not been accounted for, or that the theory needs to be modified. As discussed earlier, it is almost certain that the first of these is the explanation in an instructional lab. Any sources of error that can be identified but not easily quantified should be discussed as part of the analysis of the results

Often a theoretically accepted value for a measured physical quantity is not available. In that case it may be useful to compare the measured result from your experiment to the measured result of another experiment that measured the same quantity. In this case the % difference between the two experimental values is measured by computing the difference between the two values and dividing by the average of the two values, and multiplying by 100.

$$\%\,difference = \frac{\left| experimental\ value\#1 - experimental\ value\#2 \right|}{average\ of\ two\ exprimental\ values}(100)$$

(Again the absolute value is used since it is only the magnitude of the difference that is important, not which of the values is larger)

For example, if we measure the acceleration of gravity in two different experiments and we obtain the values of 9.75 ± 0.15 m/s^2 and 9.93 ± 0.05 m/s^2, we first compute the average of these two values:

$$average\ experimental\ value = \frac{9.75\,m/s^2 + 9.93\,m/s^2}{2} = 9.84\,m/s^2$$

Now we can find the % difference between these two values:

$$\%\,difference = \frac{\left|9.75\,m/s^2 - 9.93\,m/s^2\right|}{9.84\,m/s^2}(100) = 1.83\%$$

The per cent uncertainty for the first experiment is $\frac{0.15\,m/s^2}{9.75\,m/s^2}(100) = 1.5\%$

and the per cent uncertainty for the second experiment is $\frac{0.05\,m/s^2}{9.93\,m/s^2}(100) = .05\%$

Thus a difference of 1.5 % + .05 % = 1.6 % can be accounted for. This is smaller than the % difference computed above, so in this case the experimental uncertainty does not account for the difference and we cannot state conclusively that these two experimental results are compatible, Other sources of error not previously accounted for should be examined.

The % difference can be found for more than two experimental values if we replace the numerator in the above equation for % difference by the difference between the largest and the smallest experimentally measured value, for any size group of such measurements, and use the average of all the values as the denominator.

$$\%\,difference = \frac{largest\ experimental\ value - smallest\ experimental\ value}{average\ of\ experimental\ values}(100)$$

(The absolute value is not needed here since this numerator will always be positive.)

For example, suppose we obtain the following values of the speed of the same object from three different experiments: 2.91 ± 0.10 m/s   3.11 ± 0.050 m/s   3.04 ± 0.22 m/s

The average of these values is $\frac{2.91 m/s + 3.11\ m/s + 3.04 m/s}{3} = 3.02 m/s$

The per cent difference is $\frac{3.11 m/s - 2.91 m/s}{3.02\ m/s}(100) = 6.62\%$

The per cent uncertainties for the three values are:

$\frac{0.10 m/s}{2.91 m/s}(100) = 3.4\%$ $\frac{0.050 m/s}{3.11 m/s}(100) = 1.6\%$ $\frac{0.22 m/s}{3.04 m/s}(100) = 7.2\%$

The combined uncertainty of 3.4 % + 1.6 % + 7.2 % = 12.2 %
more than accounts for the % difference between the values, so in this case we could say that these results are compatible.

<u>Exercise 4</u>:
The result of an experiment to measure the universal gas constant is 8.24 ± 0.15 J/K mol. Calculate the per cent uncertainty and the % error for this result, and determine whether or not it is in agreement with the accepted value of 8.31 J/K mol.

If you answered 1.8 %, .842% and yes, skip to Exercise 5.
If not, continue.

$$\%\, error = \frac{8.31 J/K mol - 8.24 J/K mol}{8.31 J/K mol}(100) = .842\%$$

$$\%\, uncerta\text{int}\, y = \frac{0.15 J/K mol}{8.24 J/K mol}(100) = 1.8\%$$

The % uncertainty is larger than the % error, so the experimental result is in agreement with the accepted value.

Exercise 5:
In two different experiments, the speed of the same projectile upon leaving a particular launcher is measured to be 264.5 ± 0.5 cm/s and 271.8 ± 1.0 cm/s. Find the % difference between these two results and the % uncertainty for each and determine whether or not these two results are compatible.

If you answered 2.722 %, 0.2 %, 0.4 % and no, congratulations, you have reached the end of the tutorial. Skip to the appendix if you wish to see a derivation of the rules for propagation of uncertainty.
Otherwise continue

$$average\ experimental\ value = \frac{264.5\,cm/s + 271.8\,cm/s}{2} = 268.2\,cm/s$$

$$\%\,difference = \frac{271.8\,cm/s - 264.5\,cm/s}{268.2\,cm/s}100 = 2.722\,\%$$

$$\%\,uncertainty\ of\ 1st\ value = \frac{0.5\,cm/s}{264.5\,cm/s}(100) = 0.2\,\%$$

$$\%\,uncertainty\ of\ 2nd\ value = \frac{1.0\,cm/s}{271.8\,cm/s}(100) = 0.4\,\%$$

The combined uncertainty is 0.2 % + 0.4 % = 0.6 %.
This does not account for the 2.722 % difference, so these results are not compatible.

If you need more practice with these concepts, return to

4

If not, congratulations, you have completed the tutorial. Go to the appendix if you wish to see a derivation of the rules for propagation of uncertainty.

Appendix
Derivation of Rules for Propagation of Uncertainty

What follows is an argument for the formulas for the propagation of uncertainties through addition, subtraction, multiplication, division, raising to a power and the log function. These formulas appear on page 15 of your lab manual.[2] You can just use them when necessary, but if you have had a little calculus, you might read through the steps leading up to these equations. Another argument for the multiplication formula can be found in Swokowske (1979).

If x is a function of two variables,

$$x = f\left(y, z\right)$$

And both y and z are in turn functions of some independent parameter t.

$$y = g(t) \quad and \quad z = h(t),$$

the chain rule for multivariable calculus then allows us to write

$$\frac{dx}{dt} = \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

If we are dealing with small variations, δx, δy, δz and δt, in the quantities x, y, z and t respectively, the differentials in the above equation could be replaced to give the following approximation:

$$\frac{\delta x}{\delta t} = \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t}$$

Multiplying by δt yields

$$\delta x = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z \qquad (1)$$

[2]Cochran, William, *The Physics Lab Manual (I & II)*, page 15, Department of Physics & Astronomy, Youngstown State University, 1992

1. Propagation of Uncertainty for addition of variables:

If, in the argument above,

$$x = f\left(y,z\right) = y + z$$

Then

$$\frac{\partial f}{\partial y} = 1 \quad and \quad \frac{\partial f}{\partial z} = 1$$

and Equation 1 becomes

$$\delta x = \delta y + \delta z$$

This is the rule stated above for propagation of uncertainty through addition of variables.

2. Propagation of Uncertainty for subtraction of variables:

If, in the argument above,

$$x = f\left(y,z\right) = y - z$$

Then

$$\frac{\partial f}{\partial y} = 1 \quad and \quad \frac{\partial f}{\partial z} = -1$$

and Equation 1 becomes

$$\delta x = \delta y - \delta z$$

At this point we need to recall that the uncertainties, as we previously defined them are the absolute values of the actual variations from the average. That means that the δx, δy and δz seen here can be either positive or negative. If δy and δz in the above equation are of the same sign, the two terms in that equation will add. If they are of opposite signs, they will subtract, giving a smaller value of δx than in the previous case. Since it is the largest possible value of δx that we seek, we can use only the absolute values of δx, δy and δz and add the two terms rather than subtracting them. Thus the equation above becomes

$$\delta x = \delta y + \delta z$$

which is the same as the rule stated above for propagation of uncertainty through addition of variables.

3. Propagation of Uncertainty for multiplication of variables

$$\textit{If} \quad x = f\left(y,z\right) = \left(y\right)(z)$$

$$\textit{then} \quad \frac{\partial f}{\partial y} = z \quad \textit{and} \quad \frac{\partial f}{\partial z} = y,$$

and Equation 1 *becomes*

$$\delta x = z\,\delta y + y\,\delta z.$$

Dividing this equation by $x = yz$, *we obtain*

$$\frac{\delta x}{x} = \frac{z\,\delta y}{y\,z} + \frac{y\,\delta z}{y\,z},$$

$$\textit{or} \quad \frac{\delta x}{x} = \frac{\delta y}{y} + \frac{\delta z}{z}$$

This is the rule for propagation of uncertainty through multiplication.

4. Propagation of Uncertainty for division of variables

If $\quad x = f\left(y,z\right) = \dfrac{y}{z}$

then $\quad \dfrac{\partial f}{\partial y} = \dfrac{1}{z} \quad$ *and* $\quad \dfrac{\partial f}{\partial z} = -\dfrac{y}{z^2}$,

and Equation 1 *becomes*

$$\delta x = \left(\frac{1}{z}\right)\delta y \; + \; \left(-\frac{y}{z^2}\right)\delta z.$$

If we divide both sides by $x = \dfrac{y}{z}$, *this is the same as multiplying by* $\dfrac{z}{y}$, *so we obtain*

$$\frac{\delta x}{x} = \left(\frac{z}{y}\right)\left(\frac{1}{z}\right)\delta y \; + \left(\frac{z}{y}\right)\left(-\frac{y}{z^2}\right)\delta z.$$

or $\quad \dfrac{\delta x}{x} = \dfrac{\delta y}{y} - \dfrac{\delta z}{z}$

We can use the same argument as in the section on the subtraction rule above to eliminate the negative sign, and we see that the rule for division is the same as that for multiplication, i.e.

$$\frac{\delta x}{x} = \frac{\delta y}{y} + \frac{\delta z}{z}$$

5. Propagation of Uncertainty for a variable raised to a power

In this section and the next only two variables are involved. We can write x = f(y) and y = f(t). The analysis is the same as above except that the term containing the variable z is eliminated. So equation 1 becomes

$$\delta x = \frac{\partial f}{\partial y}\delta y \qquad (2)$$

If $\quad x = y^n,$

then $\quad \dfrac{\partial f}{\partial y} = n\,y^{\,n-1}$

and equation 2 *becomes*

$$\delta x = n\,y^{\,n-1}\,\delta y$$

Dividing by x on the left and y^n *on the right (this is valid since* $x = y^n$*), we obtain*

$$\frac{\delta x}{x} = \frac{n\,y^{\,n-1}\,\delta y}{y^n}$$

or $\quad \dfrac{\delta x}{x} = n\,\dfrac{\delta y}{y}$

As mentioned previously, δx and δy are always taken as positive, so we write

$$\frac{\delta x}{x} = |n|\frac{\delta y}{y}$$

6. Propagation of Uncertainty for the log function

$$\text{If} \qquad x = \log y,$$

$$\text{then} \quad \frac{\partial f}{\partial y} = \frac{1}{y}$$

and equation 2 *becomes*

$$\delta x = \frac{\delta y}{y}$$